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Chains of skyrmions

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Abstract: Skyrme chains are topologically-nontrivial solutions of the Skyrme model which are (quasi-)periodic in one spatial direction. We report numerical and analytic investigations which show that such solutions exist. Chains of 1-skyrmions are reasonably well approximated both as parallel vortex-antivortex pairs, and in terms of the holonomy of Yang-Mills calorons. For well-chosen periods, the chain of 1-skyrmions appears to be stable to small perturbations. However, as the period increases, the 1-skyrmions clump together, for example giving chains of 2-skyrmions or 4-skyrmions.

Keywords: Solitons Monopoles and Instantons, Sigma Models, Global Symmetries.

Contents

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1. Introduction

The Skyrme model is a three-dimensional field theory with topological soliton solutions, which is believed to provide a good model of nuclear physics; the topological solitons (skyrmions) are identified with baryons. In this paper we investigate skyrmion chains, in other words topological solutions which are periodic (or quasi-periodic) in one of the spatial directions.

Isolated skyrmions have been well-studied, up to relatively high charge [1]. Multiply-periodic configurations have also been considered, for example a "Skyrme domain wall" solution (doubly-periodic) [2], and a "Skyrme crystal" (triply-periodic) [3 – 5]. Singly-periodic chains were studied in the early days of the Skyrme model [6], but have since been neglected. There have also been studies of cylindrically-symmetric Skyrme strings [7, 8], but these are topologically trivial.

The field equations of the Skyrme model are non-linear, and no exact analytic solutions are known; so the only direct means of studying solutions is by numerical simulation. Since the system is three-dimensional, numerical methods are not fast. The most direct means of simplifying the numerics is to impose continuous symmetries on the field — this approach was used to study chains in [6]. However, if one wishes to study less symmetric fields, a number of alternative methods are known: these typically involve a relatively simple approximation, which is empirically observed to approximate the true (numerically-determined) behaviour of the solutions. The three most prominent ansätze of this type are the product ansatz (see for example [9]), the Atiyah-Manton construction [10], and the rational map ansatz [11]. We have been able to adapt the first two to study chains, but the rational map ansatz appears unsuited to this task.

The plan of the paper is as follows. Section 2 contains a discussion of the topology of skyrmion chains; section 3 has results on adapting the Atiyah-Manton construction to obtain approximate skyrmion chains from Yang-Mills calorons; section 4 discusses the approximation of skyrmion chains in terms of parallel vortex-antivortex pairs; and section 5 contains some full 3-dimensional numerical results.

We show that there is a 1-skyrmion chain with a preferred period (the period which minimizes the energy-per-period of the chain), and that the caloron-generated field is a good approximation to this solution. The vortex-antivortex pair is not as good, but becomes more accurate for denser (lower-period) chains. Generally speaking, a chain of 1-skyrmions is stable close to its preferred period, but unstable for larger periods, in the sense that the individual 1-skyrmions tend to clump into higher-charge skyrmions. We give some numerical results which illustrate this, in particular showing the appearance of chains of 2-skyrmions and of 4-skyrmions.

2. The Skyrme model and chains

The static Skyrme field U is defined on $\mathbb{R}^3$, and takes values in SU(2). Defining $L_i = U^{-1}\partial U/\partial x^i$, we take the energy density of U to be

$$\mathcal{E} := -\frac{1}{2}\mathrm{Tr}(L_i L_i) - \frac{1}{16}\mathrm{Tr}([L_i, L_j][L_i, L_j]); \tag{2.1}$$

and the normalized energy of U on $\mathbb{R}^3$ is

$$E := \frac{1}{12\pi^2}\int_{\mathbb{R}^3} \mathcal{E}\, dx^1 dx^2 dx^3. \tag{2.2}$$

The boundary condition at spatial infinity is $U \to 1$ as $r \to \infty$, where 1 denotes the identity element of SU(2). So topologically, a Skyrme field defines a map $S^3 \to \mathrm{SU}(2)$, and such a map has a degree $B \in \mathbb{Z}$. This topological charge can be computed by the integral

$$B = \int_{\mathbb{R}^3} \mathcal{B}\, dx^1\, dx^2\, dx^3, \tag{2.3}$$

where

$$\mathcal{B} = \frac{1}{24\pi^2}\epsilon_{ijk}\mathrm{Tr}(L_i L_j L_k) \tag{2.4}$$

is the topological charge density. The energy (2.2) satisfies the topological lower bound $E \geq B$. Finite-energy fields U which locally minimize E will be called skyrmions. The symmetry group of the system (including the boundary condition) consists of the spatial translations, the spatial rotations $\mathrm{O}(3)_{\mathrm{sp}}$, and the iso-rotations $\mathrm{O}(3)_{\mathrm{iso}}$.

A Skyrme chain with period $\beta > 0$ and relative orientation $R \in \mathrm{SO}(3)_{\mathrm{iso}}$ is a finite-energy Skyrme field $U : \mathbb{R}^3 \to \mathrm{SU}(2)$ satisfying

$$U(x, y, z + \beta) = R \cdot U(x, y, z), \tag{2.5}$$

minimizing the energy functional

$$E = \int_0^\beta \int_{\mathbb{R}^2} \mathcal{E}\, dx\, dy\, dz,$$

and satisfying the boundary condition $U \to 1$ as $x^2+y^2 \to \infty$. One may think of such a configuration as a chain of equally-spaced skyrmions along the z-axis, with each skyrmion being iso-rotated by R relative to its neighbours. In the appendix, it is shown that the quantity

$$B = \int_0^\beta \int_{\mathbb{R}^2} \mathcal{B}\, dx\, dy\, dz$$

is necessarily an integer, and this integer is called the *topological charge* of the chain. (If R is the identity, then it is already clear that B is an integer, since then the field U is in effect a map from $S^2 \times S^1$ to SU(2), and B is the degree of this map.)

A single skyrmion has the spherically-symmetric hedgehog form $U(\mathbf{x}) = \exp\left\{\mathrm{i}f(r)\hat{x}^j\sigma^j\right\}$, where σ^j denotes the Pauli matrices. Consequently, the iso-rotations $R \in \mathrm{SO}(3)_{\mathrm{iso}}$ of a single skyrmion may be identified with the spatial rotations, and we may think of the relative orientation of two skyrmions as being a relative orientation in physical space. If we have two well-separated 1-skyrmions, then it is well-known [9] that the force between them depends on this relative orientation:

- If the two skyrmions have the same orientation, then they repel. This case is called *aligned*.
- The strongest repulsive force occurs when one of the skyrmions is rotated by π about the line l joining them. This case is called the *repulsive channel*.
- The strongest attractive force occurs when one of the skyrmions is rotated by π about an axis perpendicular to l. This case is called the *attractive channel*.

The symmetry of such a 2-skyrmion configuration can be inferred from the dipole model of skyrmions [12], or equivalently by looking at a superposition of hedgehog configurations. A pair of skyrmions which are aligned, or in the repulsive channel, has an axial symmetry O(2) (consisting of rotations and reflections which fix l); but a pair in the attractive channel has only a discrete symmetry group D_2 (generated by reflections in two perpendicular planes whose intersection is l).

Guided by this, we define three types of unit-charge skyrmion chain as follows. Writing $U(\mathbf{x}) = \exp\{w(\mathbf{x})\}$, where w takes values in the Lie algebra $su(2)$, we may regard the relative orientation operator R defined in (2.5) as acting on $w(\mathbf{x})$ via the adjoint action. Suppose that along the z-axis, w has the form $w = g(z)v$, where v is a fixed element of the Lie algebra $su(2)$, and $g(z)$ is a real-valued function. We then identify the following three special cases:

- A chain with $R = 1$ is called *aligned*.
- A chain for which R is a rotation by π about $v \in su(2)$, is called *maximally-repulsive*.
- A chain for which R is a rotation by π about an axis perpendicular to $v \in su(2)$ is called *maximally-attractive*.

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As before, the maximally-attractive chain has only a discrete symmetry, whereas the other two have a continuous axial symmetry. We expect that only the maximally-attractive chain will have an energy less than that of an isolated skyrmion, and it is this type that we shall concentrate on in what follows.

3. Skyrme chains from calorons

The Atiyah-Manton ansatz is method of generating approximate skyrmion configurations on $\mathbb{R}^3$ by evaluating the holonomy of 4-dimensional Yang-Mills instantons. The construction is topologically natural, in the sense that the holonomy of an N-instanton is an N-skyrmion; and for an appropriate choice of the instanton scale, the resulting skyrmion field is a surprisingly good approximation to the actual skyrmion.

The method involves evaluating the holonomy of the gauge field along a family of parallel lines in the 4-dimensional space. For example, we could choose the family of lines parallel to the x^0-axis in $\mathbb{R}^4$, and then the Skyrme field is obtained via the path-ordered exponential integral

$$U(x^1, x^2, x^3) = \mathcal{P} \exp\left(\int_{-\infty}^{\infty} A_0(x) dx^0 \right).$$

In one periodic version of this construction which has been studied previously, it was shown that one may obtain a good approximation to the Skyrme crystal from instantons on the 4-torus [13].

If we want to obtain a Skyrme chain, then we should start with a gauge field on $S^1 \times \mathbb{R}^3$ satisfying

$$A_\mu(x^0 + \beta, x^1, x^2, x^3) = R \cdot A_\mu(x^0, x^1, x^2, x^3), \tag{3.1}$$

and integrate along a family of parallel lines perpendicular to the x^0-axis. Calorons provide examples of gauge fields satisfying (3.1). Let A_ν be the gauge potential of a caloron, strictly-periodic with period β, with instanton charge equal to 1, and with vanishing monopole charge (see [14, 15] for details). So near infinity, A_0 does not wind, and has the form $A_0 \approx \mathrm{i}\mu\sigma^3$, where $0 \leq \mu \leq \pi/\beta$. Now make the gauge transformation $A'_\nu = gA_\nu g^{-1} - (\partial_\nu g)g^{-1}$, where $g = \exp(\mathrm{i}\mu x^0 \sigma^3)$. In the new gauge, $A'_0 \to 0$ at infinity, and the gauge field is no longer strictly-periodic, but satisfies

$$A'_\nu(x^0 + \beta, x^1, x^2, x^3) = hA_\nu(x^0, x^1, x^2, x^3)h^{-1},$$

where $h = \exp(i\mu\beta\sigma^3)$. This gauge choice is known as the algebraic gauge. We can obtain 1-skyrmion chain configurations by computing the holonomies of such calorons in the algebraic gauge. Analytic expressions are known for all calorons with unit instanton charge and vanishing monopole charge [16, 17]; so it is feasible to study Skyrme chains using this family. Since we are mainly interested in maximally-attractive chains, we restrict attention to the calorons which yield $R^2 = 1$ but $R \neq 1$; these are the calorons with $\mu = \pi/2\beta$. By contrast, calorons with $\mu = 0$, such as the Harrington-Shepard calorons [18], give rise to aligned chains.

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ρ	$12\pi^2 E_2$	$12\pi^2 E_4$	$B-1$
0.2	10.56	509.2	2.0×10^{-4}
0.3	16.61	314.3	1.6×10^{-4}
0.4	23.20	213.7	1.1×10^{-4}
0.5	29.68	160.4	7.7×10^{-5}
0.6	35.40	135.4	6.3×10^{-5}
0.7	40.21	124.6	9.3×10^{-5}
0.8	44.25	119.6	1.9×10^{-4}

Table 1: Caloron approximation

The calorons we are interested in are symmetric under rotations about an axis in $\mathbb{R}^3$. If we choose to evaluate holonomies along lines parallel to this axis, the Skyrme chains will also have an SO(2) symmetry: these will be maximally-repulsive chains. If, on the other hand, we evaluate holonomies along lines perpendicular to the symmetry axis, we obtain maximally-attractive chains.

We have implemented the Atiyah-Manton construction for maximally-attractive chains numerically, and evaluated the energies of the resulting skyrmion chains. The family of calorons we used is parametrized by a scale parameter ρ and a period β. The caloron with scale ρ and period β is in fact a rescaling of the caloron with scale ρ/β and period 1, and since the components of the Skyrme energy behave simply under rescalings, it was sufficient only to consider calorons with fixed $\beta = 1$ and a range of values of ρ.

The holonomies were evaluated using the Runge-Kutta method. We evaluated energies in a finite box $-L \leq x, y \leq L$, and extrapolated in both the box size and the lattice spacing to obtain energies accurate to within 0.1%. We also calculated B to check the accuracy of our method. Our results are summarized in table 1.

In figure 1(a), we have plotted the minimum energy of this approximate skyrmion chain, as a function of the period β. The graph was obtained by interpolating the data in table 1 to obtain E_2 and E_4 as polynomial functions of ρ, and then minimizing the energy $E = E_2\beta + E_4/\beta$ with respect to variation in ρ. In particular, we see that the energy of these caloron-derived configurations has its lowest value $E \approx 1.16$ for period $\beta \approx 2.1$.

4. The vortex ansatz

In this section we describe an alternative ansatz for chains, which is based on the idea that a chain, especially for small period, splits into constituents. We define a Skyrme vortex to be a field of the form

$$U_v = \exp\left[\frac{1}{2}(\theta - \nu z)\mathrm{i}\sigma^3\right] \exp\left[f(r)\mathrm{i}\sigma^1\right] \exp\left[\frac{1}{2}(\theta + \nu z)\mathrm{i}\sigma^3\right], \tag{4.1}$$

where (r, θ) are polar coordinates in $\mathbb{R}^2$, and ν is a positive constant. The profile function $f(r)$ is required to satisfy the boundary conditions $f(0) = \pi/2$ and $f(r) \to 0$ as $r \to \infty$.

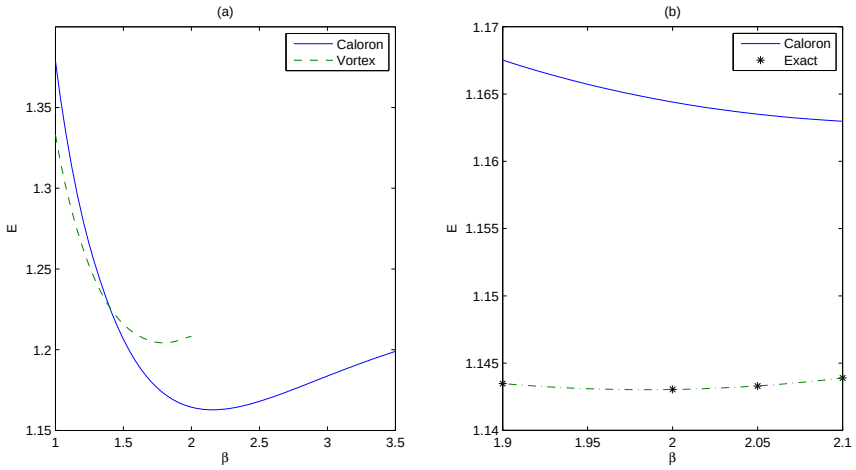

Figure 1: The energy of a 1-skyrmion chain, versus the period β, for the caloron construction (solid curve in (a) and (b)), the vortex ansatz (dashed curve in (a)), and from a full 3D calculation (dot-dash curve in (b)).

Note that U_v is smooth on $\mathbb{R}^3$, and periodic in z with period $2\pi/\nu$. The energy density of U_v is

$$\mathcal{E} = (f')^2 + \frac{\cos^2 f}{r^2} + \nu^2 \sin^2 f + (f')^2 \frac{\cos^2 f}{r^2} + (f')^2 \nu^2 \sin^2 f + \frac{\nu^2 \sin^2 f \cos^2 f}{r^2}. \tag{4.2}$$

The profile function $f(r)$ is chosen to solve the Euler-Lagrange equation for $\int_0^\infty \mathcal{E} r\, dr$. But note, for example from the second term in $\mathcal{E}$, that U_v has infinite energy per unit period. This is a consequence of the fact that U_v has the non-constant form

$$U_v \approx \exp(\mathrm{i}\theta\sigma^3) \tag{4.3}$$

as $r \to \infty$. The Skyrme vortex (4.1) has been used before, in a different context [19]: constructing vortex loops by taking finite lengths of Skyrme vortex and joining their ends together.

Notice that the field $U(x, y, z) = U_v(x, -y, -z)$ winds in the opposite direction to U_v at infinity. So we can obtain a field which is constant at infinity by taking a superposition of two vortices via the product ansatz

$$U = U_1 U_2$$

with $U_1(x, y, z) = U_v(x - a, y, z)$ and $U_2(x, y, z) = U_v(x + a, -y, -z)$, where a is a positive constant. We can obtain a similar, but more symmetric, field by using the relativized product ansatz [20]

$$U = (U_1 U_2 + U_2 U_1)/\sqrt{(\det(U_1 U_2 + U_2 U_1)}.$$

The superposition satisfies the boundary conditions of a maximally-attractive chain with period $\beta = \pi/\nu$ and $R = \mathrm{diag}(-1, -1, 1) \in \mathrm{SO}(3)$, and has the same symmetries. The

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field resembles a pair of parallel vortices separated by a distance $2a$, and its topological charge is 1.

When the separation $2a$ of the vortices is large, they attract each other, as the following heuristic argument shows. Let $C > 0$ be sufficiently large for the approximation (4.3) to be valid for $r > C$, and let a be larger than C. We can evaluate the energy of the superposition by splitting $\mathbb{R}^2$ into three regions: the two discs of radius C centred on the vortex locations, and the exterior. The energy within each disc tends to a constant as a tends to infinity. The energy in the exterior diverges as $a \to \infty$; a calculation shows that the leading contribution at large a is $4\pi\beta \ln a$. So one can reduce the energy by reducing the separation $2a$, as claimed.

So far we have not justified our choice of superposition procedure: it is important to ask whether there is another way to superpose two vortices to obtain a lower energy. Again, we have a heuristic argument why our superposition is the right thing to do, at least for large separation. Let F denote the exterior of the two discs D_1, D_2 of radius C and centres $(x, y) = (\pm a, 0)$, as before. Let $\psi : F \to \mathrm{U}(1)$ be a map such that $\psi|_{\partial D_1}$ has winding number 1 and $\psi|_{\partial D_2}$ has winding number -1. We want to minimize the energy

$$e = \beta \int_F \|\psi^{-1} d\psi\|^2 d^2x.$$

The ansatz used above corresponds to taking $\psi = \exp[\mathrm{i}(\theta_1 - \theta_2)]$, where $\theta_1(x, y)$ is the angle between $(x - a, y)$ and the x-axis, and $\theta_2(x, y)$ is the angle between $(x + a, y)$ and the x-axis. If the energy of this field is close to the true minimum, then we know our ansatz is a good one. Notice that the Skyrme term has disappeared from our energy functional; this is because the Skyrme term evaluates to zero for any U(1) field. The easiest way to find the minimum energy is to stereographically project from $\mathbb{R}^2$ to S^2; the energy e is conformally-invariant, so we are allowed to do this. The stereographic projection can be chosen so that the two circles are described by $\theta = \alpha$ and $\theta = \pi - \alpha$ in spherical coordinates $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$, where $\sin(\alpha) = C/a$. The energy functional is now written as

$$e = \beta \int_0^{2\pi} \int_\alpha^{\pi-\alpha} \left((\psi^{-1}\partial_\theta \psi)^2 + \sin^{-2}\theta (\psi^{-1}\partial_\phi \psi)^2 \right) \sin\theta \, d\theta \, d\phi.$$

A Bogomolny argument shows that this energy is minimized by $\psi(\theta, \phi) = \exp(i\phi)$; and the minimum energy is $4\pi\beta \ln \cot(\alpha/2)$. For large a, this agrees with our superposition, to leading order.

We have evaluated the energy of the superposition of two vortices for a range of values of β and a. The energies were evaluated in a finite box, and we extrapolated in the box size and the lattice spacing to obtain results accurate to within 0.1%. We tried using both the product ansatz and the relativized product ansatz, and found that the energies obtained agreed. We also evaluated the topological charge B as a check on our methods. Table 2 shows the minimum energy of the superposition, together with the value of a for which this energy is attained. The energy is plotted, as a function of the period β, in figure 1(a).

β	$12\pi^2 E$	a	$B-1$
1.0	157.9	1.13	5.4×10^{-4}
1.1	153.1	1.15	3.4×10^{-4}
1.2	149.6	1.16	3.1×10^{-4}
1.3	147.1	1.18	2.1×10^{-4}
1.4	145.2	1.20	2.0×10^{-4}
1.5	144.0	1.21	1.5×10^{-4}
1.6	143.2	1.22	1.5×10^{-4}
1.7	142.8	1.23	1.2×10^{-4}
1.8	142.6	1.24	1.1×10^{-4}
1.9	142.8	1.24	9×10^{-5}
2.0	143.1	1.25	9×10^{-5}

Table 2: Vortex approximation

5. Numerical minimization

In order to assess the approximation schemes of the previous two sections, and also to investigate the stability of the maximally-attracting 1-skyrmion chain, we implemented a full 3-dimensional numerical minimization of the energy E. The method involved a first-order finite-difference scheme for E, with the spatial points (x, y, z) being represented by a rectangular lattice having lattice spacing h. The boundary condition $U = 1$ was imposed at $|x| = L$, $|y| = L$. The energy was minimized using a conjugate-gradient method. The errors in E as a result of the finite lattice spacing h and finite size L go like h^2 and $1/L^2$ respectively, and we obtained the $h \to 0$, $L \to \infty$ limits by extrapolation. The resulting values for E have an error less than 0.05%.

The first simulation looked at the maximally-attractive 1-skyrmion chain, for each of the periods $\beta = 1.9, 2.0, 2.05, 2.1$ (since all the indications are that the preferred period $\beta_{\min}$ is close to 2). The results are plotted in figure 1(b), together with a parabola fitted to the resulting four points; we see that the preferred period is $\beta_{\min} = 1.98$, and that the minimal energy is $E_{\min} = 1.143$. The minimal configuration with period $\beta = 2$ (essentially the preferred period) is depicted in the upper row of figure 2, plotted over two periods. Subfigure (a) plots the function $\sigma = \frac{1}{2}\operatorname{tr}(U)$, or rather the surface $\sigma(\mathbf{x}) = 0$; subfigure (b) plots the charge density $\mathcal{B}$, or rather the surface $\mathcal{B}(\mathbf{x}) = 0.2 \times \max \mathcal{B}$. The same two quantities are plotted for the field derived from the caloron construction, in subfigures (c) and (d); and for the field derived from the vortex ansatz, in subfigures (e) and (f). We see that the approximate fields are qualitatively similar to the actual solution, although there are noticeable differences.

For large period, we would expect the maximally-attractive 1-skyrmion chain to be unstable to clumping, for the same reason that a finite collection of separated 1-skyrmions will clump together. For small period, however, the 1-skyrmion chain might be stable. To investigate this, we first looked at the periodic 2-skyrmion chain, where the initial

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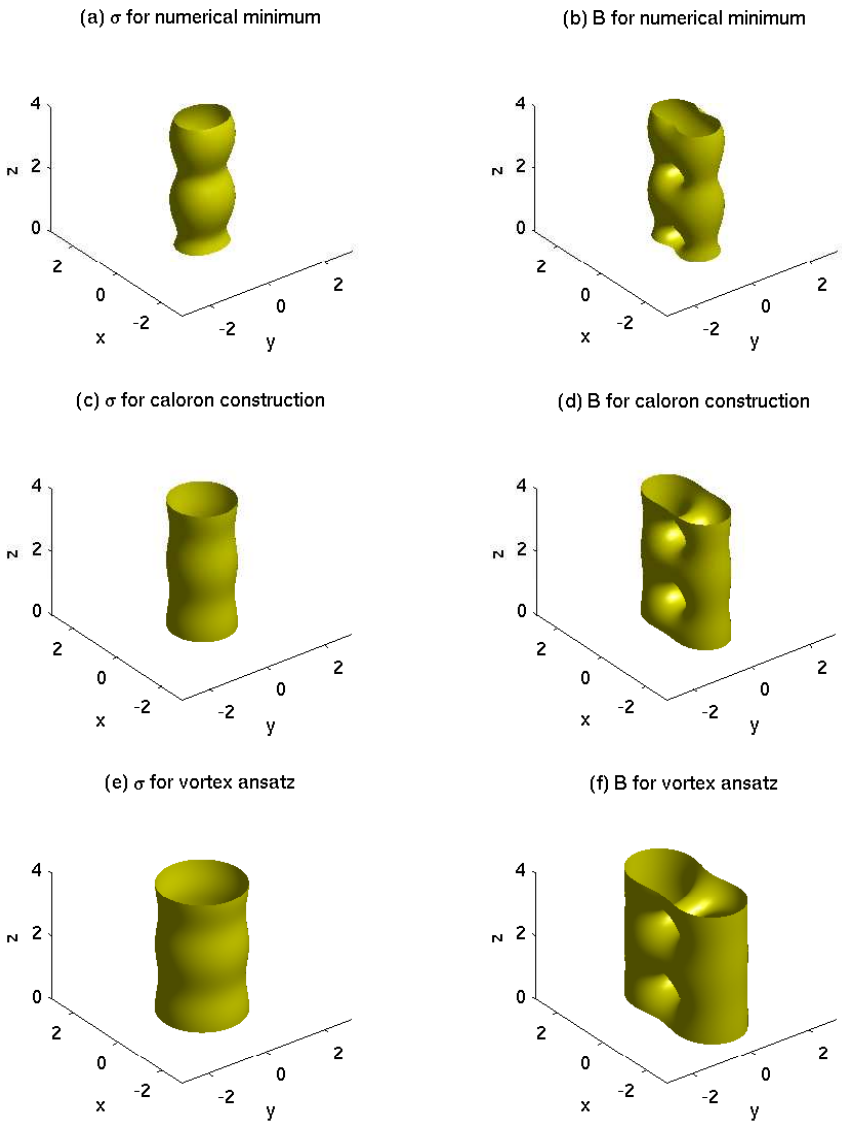

Figure 2: The field component σ and the charge density $\mathcal{B}$ for the 1-skyrmion chain with period $\beta = 2$: the numerical minimum, the caloron approximation and the vortex approximation.

configuration was taken to be (a deformation of) a pair of skyrmions on the z-axis, in the attractive channel. As before, the numerical simulation involved flowing down the energy gradient, to reach a local minimum. For period $\beta = 4$, the minimum is a pair of single skyrmions, leaving open the possiblity that the 1-skyrmion chain with period $\beta = 2$ might be stable (more on this below). But for period $\beta = 5$, the minimum is a toroidal 2-skyrmion (with axis orthogonal to the z-axis), which forms by coalescence of the two

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individual skyrmions; so the 1-skyrmion chain with period $\beta = 2.5$ is not a stable solution. The instability which leads to this coalescence is rather weak: in fact, the energy of the toroidal chain with period $\beta = 5$ is only slightly (0.1%) less than twice the energy of a 1-skyrmion chain with period 2.5.

One can repeat this sort of investigation, looking at skyrmion chains with higher charge. There is evidence suggesting that a 4-skyrmion chain (chain of α-particles) might be particularly favourable: finite-length chains of α-particles occur in $\mathbb{R}^3$, at least when one modifies the system by adding a significant pion mass [21 – 23]. To further investigate the stability of the 1-skyrmion chain, we looked at the periodic 4-skyrmion chain with period $\beta = 8$, where the initial configuration was taken to be the 1-skyrmion chain over four periods, deformed by (i) moving the four skyrmions slightly towards one another, (ii) moving each of them slightly off the z-axis (in different directions), and (iii) adding a random perturbation to the field. Minimizing from this starting-point yielded the original 1-skyrmion chain. We hypothesize, therefore, that the 1-skyrmion chain with its preferred period is a stable solution — in other words, a local minimum of the static energy functional. The results of our numerical simulations are consistent with this, but do not, of course, prove it.

The local minimum referred to in the previous paragraph is certainly not a global minimum. This can, for example, be seen by using a different initial configuration, designed to favour a chain of α-particles. For this final simulation, we used a periodicity condition which amounts to rotating neighbouring α-particles by π about the periodic axis [23]; this is of the form (2.5) with $R^2 = 1$ and $R \neq 1$. (For well-separated α-particles, as in figure 3(a), imposing strict periodicity makes very little difference). The α-particle chain with period $\beta = 8$ has has significantly (4%) lower energy than four times the energy of a 1-skyrmion with period 2; the plot of its charge density in figure 3(a) shows that, for this value of the period, the α-particles are quite well localized. One can lower its energy even further by reducing the period, and so allowing the α-particles to move closer together: in figure 3(b), we see the α-particle chain with period $\beta = 3$, and its energy is 6% lower than the minimal value ($E = 1.143$) for a 1-skyrmion chain. The likelihood is that, for any given period, there are many local minima of the energy, corresponding to B-skyrmion chains for various values of the topological charge B.

6. Concluding remarks

From figures 1 and 2 we see that the Atiyah-Manton construction gives a reasonably good approximation to the minimal-energy 1-skyrmion chain, with energy only 1% above its true value. The vortex-antivortex approximation is not quite as good, but does emphasize the constituent structure of skyrmion chains. It appears to be a common feature of topological soliton systems that soliton chains exhibit a constituent structure when the soliton size is comparable to the period: the solitons fragment into fractional-charge objects, and this fragmentation occurs in a transverse direction, so that rotational symmetry about the chain axis is lost. These features are apparent from figure 2(b), where one sees evidence of the parallel vortex-antivortex pair. For low-period chains (more precisely, where the

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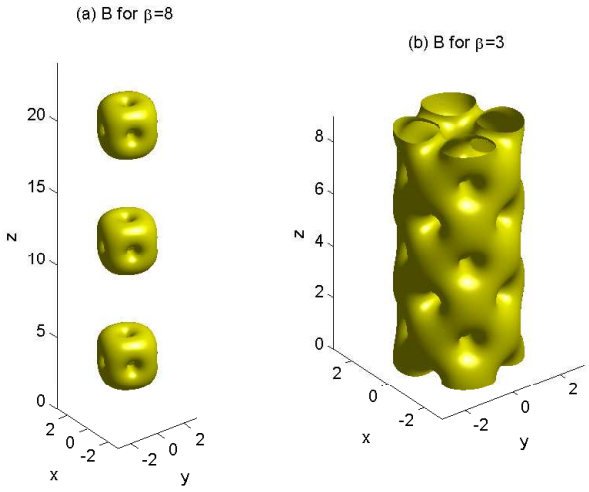

Figure 3: The charge density $\mathcal{B}$ for 4-skyrmion chains with periods $\beta = 8$ and $\beta = 3$, over three periods.

period is small compared to the natural soliton size), one expects the chain to resemble a vortex pair of this type.

The Skyrme crystal [3 – 5] has an energy-per-baryon of $E = 1.036$, and one could construct a skyrmion chain by cutting a chain out of this crystal, in other words by truncating it in the x- and y-directions. A very thick chain obtained in this way should have rather low energy (becoming lower as the chain became thicker). Among thin chains (with transverse size comparable to the natural soliton size), the chain of α-particles may well be the lowest-energy solution. The preliminary calculation reported in the previous section showed that its energy is $E \approx 1.07$ for period $\beta = 3$. The chain of 1-skyrmions has a much higher energy-per-baryon of $E = 1.143$, but still seems to be stable to small perturbations. It would be worth making a more comprehensive study of N-skyrmion chains for various N, for a wide range of periods β, and with various periodicity conditions; but that will require more intensive computational effort than we have used in deriving the results reported here.

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A. Topology of chains

In this appendix, we show that the topological charge B is an integer, even when the field is not strictly-periodic. It is a special case of the following generalisation of the degree theorem, which appears to be new.

Theorem. *Let Σ be an n-dimensional compact manifold without boundary, with volume form ω, and with $H^{n-1}(\Sigma) = 0$ and $H^n(\Sigma) = \mathbb{Z}$. Let M be an $(n-1)$-dimensional manifold such that $H^n(M \times S^1) = \mathbb{Z}$. Suppose that $\mathrm{SO}(2)$ acts on Σ and that ω is $\mathrm{SO}(2)$-invariant. Fix an element $\sigma \in \mathrm{SO}(2)$ and let $\phi : M \times \mathbb{R} \to \Sigma$ be a map satisfying*

$$\phi(x, y + \beta) = \sigma\phi(x, y) \; \forall x \in M, y \in \mathbb{R}.$$

Then ϕ has an integer degree, computed by the integral

$$deg(\phi) = \frac{1}{Vol(\Sigma)} \int_0^\beta \int_M \phi^*\omega.$$

Furthermore, $deg(\phi)$ is independent of the choice of $\mathrm{SO}(2)$-invariant volume form ω.

Proof. The idea of the proof is simple: we deform ϕ to a strictly-periodic map using the SO(2) action, and show that the integral is unchanged by this deformation. First, we introduce some notation: we write the SO(2) action as $R_s : \Sigma \to \Sigma$, with $s \in \mathbb{R}/\mathbb{Z} \cong \mathrm{SO}(2)$. Let $X \in T\Sigma$ be the associated vector field. Let $t_0 \in \mathbb{R}/\mathbb{Z}$ be such that $R_{t_0} = \sigma^{-1}$, and let $t(x, y) = t_0 y/\beta$ be a function on $M \times \mathbb{R}$. We define a deformation

$$\tilde{\phi}(x, y) = R_{t(x,y)}(\phi(x, y)), \; x \in M, y \in \mathbb{R}.$$

Then $\tilde{\phi}$ is a strictly periodic map, hence has a degree computed by

$$deg(\tilde{\phi}) = \frac{1}{Vol(\Sigma)} \int_0^\beta \int_M \tilde{\phi}^*\omega.$$

Now we show that

$$\int_0^\beta \int_M \tilde{\phi}^*\omega = \int_0^\beta \int_M \phi^*\omega.$$

For any form $\theta \in \Lambda^*\Sigma$, one can show that

$$\tilde{\phi}^*\theta = \phi^* R_t^*\theta + \phi^*(i_X R_t^*\theta) \wedge dt.$$

Here i_X denotes the inner derivative of a form. In the particular case $\theta = \omega$, one has $R_t^*\omega = \omega$ (because the volume form is SO(2)-invariant). Hence

$$\tilde{\phi}^*\omega = \phi^*\omega + \phi^*(i_X\omega) \wedge dt.$$

By Cartan's formula, we have

$$L_X\omega = i_X d\omega + d i_X\omega,$$

where L_X denotes the Lie derivative. We see immediately that $d\omega = 0$, because $\omega \in \Lambda^n\Sigma$. On the other hand, $L_X\omega$ must vanish since ω is SO(2)-invariant. It follows that $i_X\omega$ is closed. Since $H^{n-1}(\Sigma) = 0$, $i_X\omega$ is exact, in other words, there exists a $\mu \in \Lambda^{n-2}\Sigma$ such that $i_X\omega = d\mu$. Therefore

$$\begin{aligned} \tilde{\phi}^*\omega &= \phi^*\omega + \phi^*(d\mu) \wedge dt \\ &= \phi^*\omega + d(\phi^*\mu \wedge dt). \end{aligned}$$

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Integrating and applying Stoke's theorem, we obtain

$$\int_0^\beta \int_M \tilde{\phi}^* \omega = \int_0^\beta \int_M \phi^* \omega + \left[\int_M \phi^* \mu \wedge dt \right]_0^\beta .$$

The boundary term vanishes since t is constant on the domain of integration, so we have the desired result.

That $deg(\phi)$ is independent of the choice of volume form follows from the corresponding property of the classical degree. If ω' is any other SO(2)-invariant volume form, then

$$\begin{aligned} \int_0^\beta \int_M \phi^* \omega' &= \int_0^\beta \int_M \tilde{\phi}^* \omega' \\ &= \int_0^\beta \int_M \tilde{\phi}^* \omega \\ &= \int_0^\beta \int_M \phi^* \omega . \end{aligned}$$

References

[1] R.A. Battye and P.M. Sutcliffe, *Skyrmions, fullerenes and rational maps*, *Rev. Math. Phys* **14** (2002) 29 [hep-th/0103026].

[2] R.A. Battye and P.M. Sutcliffe, *A Skyrme lattice with hexagonal symmetry*, *Phys. Lett.* **B 416** (1998) 385 [hep-th/9709221].

[3] M. Kugler and S. Shtrikman, *A new Skyrmion crystal*, *Phys. Lett.* **B 208** (1988) 491.

[4] M Kugler and S Shtrikman, *Skyrmion crystals and their symmetries*, *Phys. Rev.* **D 40** (1989) 3421.

[5] L. Castillejo, P.S.J. Jones, A.D. Jackson, J.J.M. Verbaarschot and A. Jackson, *Dense Skyrmion systems*, *Nucl. Phys.* **A 501** (1989) 801.

[6] I. Zahed, A. Wirzba, U.G. Meissner, C.J. Pethick and J. Ambjørn, *Periodic Skyrme solitons*, *Phys. Rev.* **D 31** (1985) 1114.

[7] A. Jackson, *String like solutions in the Skyrme model*, *Nucl. Phys.* **A 493** (1989) 365.

[8] M. Nitta and N. Shiiki, *Skyrme strings*, *Prog. Theor. Phys.* **119** (2008) 829 [arXiv:0706.0316].

[9] B.J. Schroers, *Dynamics of moving and spinning Skyrmions*, *Z. Physik* **C 61** (1994) 479 [hep-ph/9308236].

[10] M.F. Atiyah and N.S. Manton, *Skyrmions from instantons*, *Phys. Lett.* **B 222** (1989) 438.

[11] C.J. Houghton, N.S. Manton and P.M. Sutcliffe, *Rational maps, monopoles and Skyrmions*, *Nucl. Phys.* **B 510** (1998) 507 [hep-th/9705151].

[12] N.S. Manton and P.M. Sutcliffe, *Topological solitons*, Cambridge University Press, Cambridge U.K. (2004).

[13] N.S. Manton and P.M. Sutcliffe, *Skyrme crystal from a twisted instanton on a four torus*, *Phys. Lett.* **B 342** (1995) 196 [hep-th/9409182].

JHEP12(2008)093

[14] D.J. Gross, R.D. Pisarski and L.G. Yaffe, *QCD and instantons at finite temperature*, *Rev. Mod. Phys.* **53** (1981) 43.

[15] T.M.W. Nye, *The geometry of calorons*, PhD thesis, Edinburgh U.K. hep-th/0311215.

[16] T.C. Kraan and P. van Baal, *Periodic instantons with non-trivial holonomy*, *Nucl. Phys.* **B 533** (1998) 627 [hep-th/9805168].

[17] K.-M. Lee and C.-H. Lu, SU(2) *calorons and magnetic monopoles*, *Phys. Rev.* **D 58** (1998) 025011 [hep-th/9802108].

[18] B.J. Harrington and H.K. Shepard, *Periodic Euclidean solutions and the finite temperature Yang-Mills gas*, *Phys. Rev.* **D 17** (1978) 2122.

[19] V.B. Kopeliovich and B.E. Stern, *Exotic Skyrmions*, *JETP Lett.* **45** (1987) 203 [*Pisma Zh. Eksp. Teor. Fiz.* **45** (1987) 165].

[20] E.M. Nyman and D.O. Riska, *Symmetrized product ansatz in the Skyrme model*, *Phys. Lett.* **B 203** (1988) 13.

[21] R. Battye and P. Sutcliffe, *Skyrmions and the pion mass*, *Nucl. Phys.* **B 705** (2005) 384 [hep-ph/0410157].

[22] R. Battye and P. Sutcliffe, *Skyrmions with massive pions*, *Phys. Rev.* **C 73** (2006) 055205 [hep-th/0602220].

[23] R. Battye, N.S. Manton and P. Sutcliffe, *Skyrmions and the α-particle model of nuclei*, *Proc. Roy. Soc. Lond.* **A 463** (2007) 261 [hep-th/0605284].

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